boundary-layer parameters were calculated from this data applying a modified variable step size Simpson's rule using a high-speed computer. This data was then used to calculate the friction velocity using the Ludwieg-Tillman empirical relation. These values, together with the friction velocity obtained by a method suggested by Bradshaw,[11] are also given in the table. The good agreement in the values of friction velocity obtained by various methods further demonstrates the care taken in doing the experiments. The velocity profile data was plotted in "law of the wall" coordinates to check the existence of log law in the overlap region. Also, good agreement with the Clauser's[12] curve amply proved that the boundary layers were fully-developed, zero pressure gradient, two-dimensional, and turbulent. The new device is thus expected to be useful for skin-friction measurements, specially in three-dimensional flows.

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Adjustment to the Shape Factor Reflection Analysis

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Nomenclature

S = diffuse component shape factor
T = specular component shape factor
V_o = detecting system dark level
V_D = nonregularly reflected component
V_s = regularly reflected component
V_1 = nonregular reflection amplitude component
V_2 = regular reflection amplitude component
β = total scatter angle = $\psi + \theta$
θ = angle of reflection
ψ = angle of incidence

Superscripts
D = diffuse
s = specular

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Introduction

An empirical expression, presented by the author,[1–3] closely represents the radiant power reflected from surfaces. In Ref. 1, the angular variation of "radiation diffusely reflected" from flat powder aluminum oxide samples was examined. This actual angular variation was not compatible with the ideal diffuse assumption. By introducing a shaping factor, an improved compatibility was obtained. The samples discussed in Ref. 2 were made of brass and were first polished and then roughened to various degrees by glass shot peening. Because these samples exhibited vastly divergent reflection characteristics—some samples were essentially mirrors—the empirical formulation presented in Ref. 1 was insufficient. The author, therefore, deduced the expression, which consists of two shape factor expressions.

Based on the assumption that the regular and nonregular reflection components are summable, this expression consists of a near diffuse portion (nonregularly reflected), which is directly dependent on a shape factor S, and a "glare" or near specular portion (regularly reflected), which is dependent on a shape factor T. The first term corresponds to the nonregularly reflected component that was presented in Ref. 1 and augmented for a particular illumination-detection condition. The second term corresponds to the regularly reflected component. In both Refs. 1 and 2, data were acquired by varying both the angle of incidence and the angle of reflection such that their sum was constant. Both of these angles are measured in the plane of incidence, and the procedure is accomplished by rotating the surface.

To foster additional confidence in the use of the empirical expression, the author acquired additional data in the "usual way" and presented them in Ref. 3. The samples used were the same samples that were used in Ref. 2, and the "usual way" in which the author acquired the data implies that the angle of incidence was held constant while the reflection angle was varied. The empirical expression represents the data quite well. Thus, the angular variations as represented by the expression appear to be correct, because the data, acquired by two different techniques (independent variables), were fit by the expression.

To test the formulation further, the author studied several types of paper, whose surface properties are intermediate between the "diffuse," aluminum oxide surfaces of Ref. 1 and the "roughened mirror," brass surfaces of Refs. 2 and 3. Though each set of acquired data differed slightly, characterized by the type of paper, all the sets indicated upon data reduction that the second term in the empirical expression is inappropriate.

This Note presents the bidirectional reflectance data, typical of most of the papers that were used to correct the expression, particularly, its regular component portion.

Analysis

The empirical expression presented by the author[2,3] is

$$\begin{aligned} V &= V_o + V_D \cos\psi + V_s \cos\psi \cos(\beta - \psi), \qquad o \leq \theta < \theta_m \\ &= V_o + V_D(A/B)\cos(\beta-\psi) + V_s \cos\psi \cos(\beta-\psi) \qquad \theta_m \leq \theta \leq \pi/2 \end{aligned} \tag{1}$$

in which

$$V_D = V_1[\cos\psi \cos(\beta-\psi)/\cos^2 \beta/2]^{S-1}$$
and
$$V_s = V_2[\cos\psi \cos(\beta-\psi)/\cos^2 \beta/2]^{T-1} \tag{2}$$

The parameter θ_m, which can be defined by referring to Fig. 1, is the value of θ such that A' and B' are equal and is the change-over point from the condition of over-illumination [Eq.

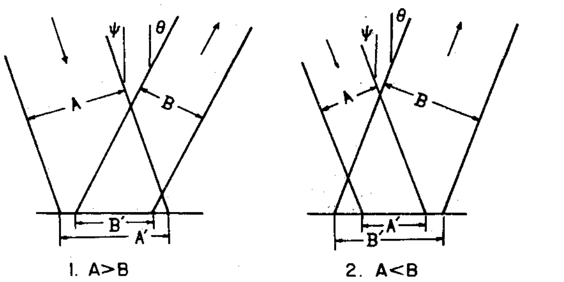

Fig. 1 Illustration of illumination-detection conditions.

(1)] to over-detection [Eq. (2)]. In this representation, V_D represents the nonregularly reflected component and, when $S = 1$, reduces to the diffuse case, i.e., a constant. The parameter V_s represents the regularly reflected component and approaches a Kronecker delta function[4] at $\psi = \beta/2$ for T very large. By the very nature of these two components, T is always assumed to be larger than S. Thus, the received radiant power can be represented by

$$V = V_o + V_1 G^{S-1} g_1 + V_2 G^{T-1} g_2 \tag{3}$$

in which

$$G = \cos\psi \cos(\beta - \psi)/\cos^2(\beta/2)$$
$$g_1 = \cos\psi, \qquad \theta < \theta_m$$
$$g_1 = (A/B)\cos(\beta - \psi), \qquad \theta \geqq \theta_m$$
$$g_2 = \cos\psi \cos(\beta - \psi)$$

Equation (3) is based on the usual assumption that the two components of the received radiant power that are reflected are summable.

As mentioned in Refs. 2 and 3, Eq. (3) fits the bidirectional data, descriptive of the radiant power reflected from roughened, polished brass, to within 10% over a wide range of laboratory variables, i.e., σ, ψ, θ, and β. When the data[2,3] involving the power reflected from a sample which was neither ideally diffuse or specular were reduced, T was found to be very large, V_2 very small, V_1 of moderate size, and S greater than one. For the data representing the typical reflectance characteristics of a "shiny" brass sample, S was found to be approximately one, V_1 small, and V_2 and T large. Situations, in which S, T, V_1, and V_2 are all of moderate size, remained to be investigated.

To accomplish this, experimental data were acquired by using both the constant β and the constant ψ techniques. The samples were constructed of filter paper. The solid curves of Figs. 2 and 3 illustrate the raw data for both data acquisition techniques.

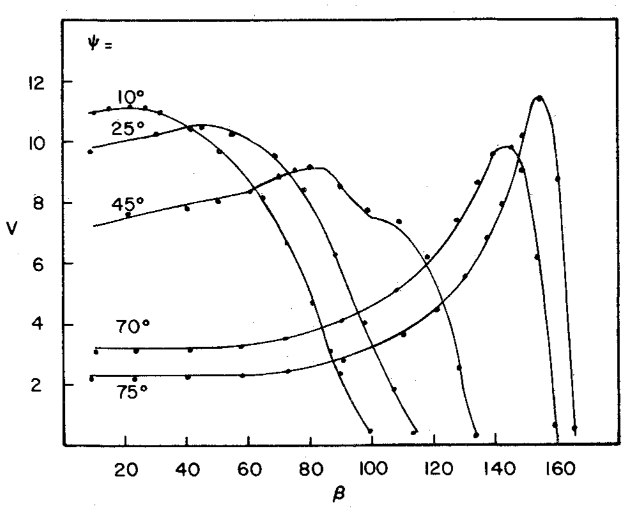

Fig. 2 Data from white filter paper; reflected power vs scatter angle $\beta (= \theta + \psi)$: ...from Eq. (3), —raw data.

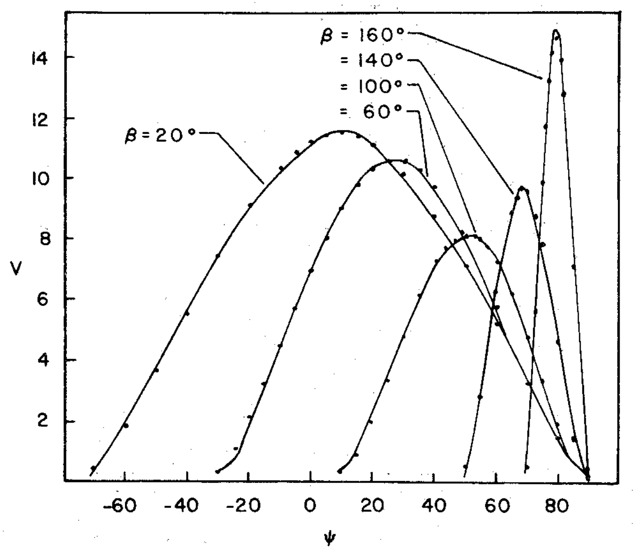

Fig. 3 Data from white filter paper; reflected power vs angle of incidence, ψ: ...from Eq. (3), —raw data.

When the same data reduction procedure discussed in Refs. 2 and 3 was used, a systematic discrepancy was observed. The error occurred in the position of "the specular peak" predicted by Eq. (3) and was particularly evident in the ψ constant data. The error was determined by calculating the derivative of the "specular" portion of Eq. (3) with respect to β. The result is either

$$\frac{T-1}{T} = \tan(\beta - \psi)\cot\beta/2$$

or

$$\tan\psi = \tan\frac{\beta}{2}\left\{1 + \frac{1}{T} + \frac{2}{T^2}\sin^2\frac{\beta}{2}\sum_{n=o}^{\infty}\left[\frac{2\sin^2\beta/2}{T}\right]^n\right\} \tag{4}$$

As can be seen from Eq. (4) in the case of T large, $\psi = \beta/2$, which should be the case, but if T were of moderate size, a large error is possible.

Improved Analysis

The above cited discrepancy indicates the need for an improvement of the second portion of Eq. (3). From purely a heuristic standpoint, this can be obtained by allowing $g_2 = 1$ in Eq. (3). In this case, the derivative of the specular portion of Eq. (3) yields ψ equals $\beta/2$ regardless of the value of T.

This small change, when incorporated in the data reduction procedure,[2,3] greatly enhances the fit of Eq. (3) to the raw data. Figures 2 and 3 may be used to compare this compatibility. Figure 2 represents the ψ-constant data acquisition situation. The difference between the points computed, when the corrected form of Eq. (3) and the raw data curve are used, is quite small. Figure 3 represents the β constant data acquisition situation for various values of β. The difference between the raw data curve and points computed by the corrected Eq. (3) is again quite small except when $\beta = 160°$. This exception is about 5%.

Summary

An empirical expression introduced by the author[1] was modified to represent the reflected radiant energy from surfaces that were far from being diffuse.[2,3] This expression fit the data very well. A need arose to adjust the expression for flat surfaces that are intermediate between near diffuse and near specular; a characteristic possessed by the surfaces of paper. When radiant power (in the visible portion of the spectrum) is near normal incidence to a paper surface, the reflected power is very nearly "diffuse." When this power is at only moderate angles, a large reflected "glare" is apparent. After the expression had been

adjusted for the additional data, the correct formulation compared very well with the raw bidirectional reflectance data.

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Note on Rectangular Finite Elements with In-Plane Forces

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QUADRILATERAL elements with linear boundaries are frequently used in solving two-dimensional elasticity problems by the finite element method. Although it is possible to generate element matrices with progressively higher-order approximation, the lower-order models are widely used in practice. For a four-node quadrilateral, fairly simple element matrices are commonly derived utilizing any of the following bases: 1) A composite of four constant strain triangles with static condensation of the centroidal node.[2,4] 2) Bilinear shape functions in the isoparametric coordinates which leads to linearly varying strain state for rectangular or parallelogram elements.[1,3,5,6] 3) As in Basis 2) with additional parabolic modes associated with free parameters which are finally reduced out.[5]

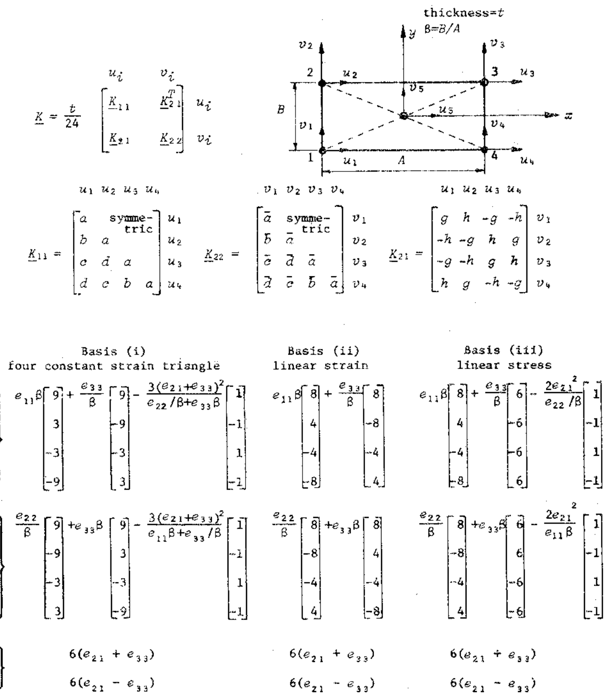

Fig. 1 Stiffness coefficients for a rectangle element.

Under Bases 1) and 2), inter-element compatibility of displacement is maintained, whereas under Basis 3) it is violated. In everyday usage of finite element programs, rectangular elements (Fig. 1) are frequently used, and it is the principal objective here to point out certain special characteristics of the element matrices for such elements.

Using fairly simple algebra, explicit expressions for the reduced stiffness matrix under Basis 1) could be obtained as shown in Turner et al.[4] Under Basis 2), the stiffness coefficients are already available.[1] The modifications needed due to the addition of the parabolic modes in Basis 3) could also be worked out explicitly with some effort. Assuming zero reference temperature and an orthotropic stress-strain relation expressed by Eq. (1), the needed expressions for the stiffness coefficients are summarized in Fig. 1.

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \begin{bmatrix} e_{11} & e_{21} & 0\\ e_{21} & e_{22} & 0\\ 0 & 0 & e_{33}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{bmatrix} - T\begin{bmatrix}\gamma_{11}\\ \gamma_{22}\\ 0\end{bmatrix} \tag{1}$$

By taking the difference between the expressions shown in Fig. 1, the following interesting relations between the matrices obtained under different bases [indicated as superscripts in Eq. (2)] could be written.

$$\begin{aligned}
\mathbf{K}_{11}^{(i)} - \mathbf{K}_{11}^{(ii)} &= [e_{11}\beta + e_{33}/\beta - 3(e_{21}+e_{33})^2/(e_{22}/\beta + e_{33}\beta)]\mathbf{N}\\
\mathbf{K}_{22}^{(i)} - \mathbf{K}_{22}^{(ii)} &= [e_{22}/\beta + e_{33}\beta - 3(e_{21}+e_{33})^2/(e_{11}\beta + e_{33}/\beta)]\mathbf{N}\\
\mathbf{K}_{11}^{(ii)} - \mathbf{K}_{11}^{(iii)} &= (2e_{33}/\beta + 2e_{21}^2/e_{22}/\beta)\mathbf{N}\\
\mathbf{K}_{22}^{(ii)} - \mathbf{K}_{22}^{(iii)} &= (2e_{33}\beta + 2e_{21}^2/e_{11}\beta)\mathbf{N}
\end{aligned} \tag{2}$$

where

$$\mathbf{N} = \begin{bmatrix} 1 & \text{symmetric} & & \\ -1 & 1 & & \\ 1 & -1 & 1 & \\ -1 & 1 & -1 & 1\end{bmatrix}$$

and the cross coefficient matrix $\mathbf{K}_{21}$ is the same under all bases, independent of the aspect ratio β, and is purely a function of the elasticity constants e_{21} and e_{33}.

Assuming a constant temperature T within an element, the thermal load vector $\mathbf{P}$ for an element could be expressed as

$$\mathbf{P} = \frac{t\cdot T}{2}[\overset{u_1}{-\gamma_{11}A} \quad \overset{u_2}{-\gamma_{11}A} \quad \overset{u_3}{\gamma_{11}A} \quad \overset{u_4}{\gamma_{11}A} \quad \overset{v_1}{-\gamma_{22}B} \quad \overset{v_2}{\gamma_{22}B} \quad \overset{v_3}{\gamma_{22}B} \quad \overset{v_4}{-\gamma_{22}B}] \tag{3}$$

and is independent of the bases considered here. It may be mentioned here that the reduced stiffness matrix derived under Basis 3) or the one derived in Przemieniecki[3] and Turner et al.[4] using linear-stress assumption leads to identical results for rectangular elements.

The knowledge of the structural similarities between the different matrices as outlined here may be fruitfully utilized in a programing code. Without much effort, it would be possible to incorporate the different formulation bases in a single package, offering a variety in the choice of the elements. For a certain class of problems permitting a regular element mesh, the full power and arbitrariness of the finite element method is not needed. With a suitable node numbering system and using the coefficients in Fig. 1, the equations for the global stiffness matrix could be written explicitly as a set of recursive difference equations, much like the way finite difference operator equations are written. Analytical techniques for solving recursive difference equations are already available. These could be fruitfully utilized in making parametric investigations of some problems, which, at present, require numerous computer runs using a general-purpose finite element code.

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